13. Gnedenko B. V., Course of the Theory of Probability., ed. 2, M., Gostekhizdat, 1953.
14. Gradshtein I. S. and Ryzhik I. M., Tables of Integrals, Sums, Series and Products., M., Fizmatgiz, 1963.
15. Bronskii A. P., The occurence of the after-effect in a solid. PMM Vol. 5, No. 1, 1941.
16. Slonimskii G. L., On the law of deformation in the real materials. I. Zh. tekhn. fiz., Vol. 9, No. 20, 1939.
17. Koltunov M. A., Influence functions in the theory of shells with persistence properties. Coll. Studies in the Theory of Plates and Shells., Vol. 5, Kazan', Izd. Kazansk. Univ. 1967.
18. Gromov V. G., Kalinin V. A., Stepanenko Iu. P. and Shliafman E. M., Approximation of the experimental data obtained in investigating the viscoelastic properties of polymers., from the book: Abstracts from reports of the 3rd All-Union Symposium of the Theoretical and Applied Mechanics., M., "Nauka", 1968.
19. Stepanenko Iu. P., Problems encountered during the experimental investigation on the viscoelastic properties of polymers., Tr. Rostovsk. Univ. inzh. zh-d. transp., M., "Transport", No. 69, 1967.

Translated by L. K.

ON CERTAIN EXACT SOLUTIONS OF THE FOURIER EQUATION FOR REGIONS VARYING WITH TIME

PMM Vol. 35, No. 4, 1971, pp. 759-760

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(Received January 11, 1971)

1. Let us use the method given in [1] to solve the first boundary value problem for the three-dimensional Fourier equation in Cartesian coordinates

$$\sum_{i=1}^{n} \frac{\partial^2 U}{\partial x_i^2} = \frac{\partial U}{\partial t} + f(x_1, \ldots, x_n, t) \qquad (n = 1, 2, 3) \tag{1.1}$$

defined on the domain bounded by coordinate planes moving in accordance with some rules $R_i^{(0)}(t)$ and $R_i^{(1)}(t)$ so that

$$x_i \in (R_i^{(0)}(t), R_i^{(1)}(t))$$

where i denote the coordinate axis number. Assume that the functions $R_i^{(0)}$ and $R_i^{(1)}$ possess continuous first and second order derivatives. We then obtain

$$\sum_{i=1}^{n} \frac{1}{\eta_i^2} \left[\frac{\partial^2 V}{\partial y_i^2} + \frac{1}{4} (\eta_i^3 \eta_i^{\cdot\cdot} y_i^2 + 2\eta_i^3 R_i^{(0)\cdot\cdot} y_i) V\right] = \frac{\partial V}{\partial t} + \frac{f}{q} \tag{1.2}$$

$$U = q(y_1, \ldots, y_n, t) V(y_1, \ldots, y_n, t), \qquad y_i = \frac{x_i - R_i^{(0)}}{\eta_i} \qquad (\eta_i = R_i^{(1)} - R_i^{(0)}) \tag{1.3}$$

$$q = \prod_{i=1}^{n} \frac{1}{\sqrt{\eta_i}} \exp\left[-\frac{1}{4}\left(\eta_i\eta_i^{\cdot} y_i^2 + 2\eta_i R_i^{(0)\cdot} y_i + \int (R_i^{(0)\cdot})^2 dt\right], \qquad y_i \in (0, 1)$$

Equation (1.2) admits an exact solution in the terms of well-known functions, provided that the following conditions hold simultaneously for all i :

$$\eta_i^3\eta_i^{\cdot\cdot} = \text{const}, \qquad \eta_i^3 R_i^{(0)\cdot\cdot} = \text{const}$$

if

(1)
$$\eta_i = \text{const}, \qquad R_i^{(0)} = M_i t^2 + A_i t + B_i \tag{1.4}$$

or

(2)
$$\eta_i = \sqrt{M_i t^2 + A_i t + B_i}, \qquad R_i^{(0)} = C_i\sqrt{M_i t^2 + A_i t + B_i} + D_i t + E_i \tag{1.5}$$

or

(3)
$$\eta_i = A_i t + B_i, \qquad R_i^{(0)} = \frac{C_i}{A_i t + B_i} + D_i t + E_i \tag{1.6}$$

Here A_i, B_i, C_i, D_i, E_i, M_i are constants depending on i. Particular cases of the motions represented by (1.4) and (1.5) have been obtained in [1 - 3].

2. If the domain is given in the cylindrical coordinate system and is bounded by planes in the z-direction and by surfaces moving according to the rules

$$z = R_3^{(0)}(t), \quad z = R_3^{(1)}(t); \qquad r = R_1(t), \quad r = R_2(t) \equiv \alpha R_1(t) \quad (\alpha = \text{const})$$

in the r-direction, then the proposed method transforms the Fourier equation in U into the following equation written in terms of another function V:

$$\frac{1}{R_1^2}\left[\frac{1}{y_1}\frac{\partial}{\partial y_1} y_1 \frac{\partial V}{\partial y_1} + \frac{1}{4} y_1^2 R_1^3 R_1^{\cdot\cdot} V + \frac{1}{y_1^2}\frac{\partial^2 V}{\partial \varphi^2}\right] +$$
$$+ \frac{1}{\eta^2}\left[\frac{\partial^2 V}{\partial y_3^2} + \frac{1}{4}(\eta^3\eta^{\cdot\cdot} y_3^2 + 2\eta^3 R_3^{(0)\cdot\cdot} y_3)V\right] = \frac{\partial V}{\partial t} + \frac{f(y_1 R_1, \varphi, y_3\eta, t)}{q} \tag{2.1}$$

where

$$U = q(y_1, y_3, t)\, V(y_1, \varphi, y_3, t) \tag{2.2}$$

$$y_1 = \frac{r}{R_1} \quad (y_1 \in (\alpha, 1)), \qquad y_3 = \frac{x_3 - R_3^{(0)}}{\eta} \quad (\eta = R_3^{(1)} - R_3^{(0)})$$

$$q = \frac{1}{R_1\sqrt{\eta}} \exp\left[-\frac{1}{4}\left(y_1^2 R_1 R_1^{\cdot} + \eta\eta^{\cdot} y_3^2 + 2\eta R_3^{(0)\cdot} y_3 + \int (R_3^{(0)\cdot})^2 dt\right)\right]$$

Equation (2.1) admits an exact solution in the terms of well-known functions provided that the conditions

$$R_1^3 R_1^{\cdot\cdot} = \text{const}, \qquad \eta^3\eta^{\cdot\cdot} = \text{const}, \qquad \eta^3 R_3^{(0)\cdot\cdot} = \text{const} \tag{2.3}$$

hold simultaneously when

$$R_1 = \sqrt{M_1 t^2 + A_1 t + B_1}, \qquad R_2 = \alpha R_1 \qquad (\alpha,\ A_1,\ B_1,\ M_1 = \text{const}) \tag{2.4}$$

while $R_3^{(0)}$ and $R_3^{(1)}$ satisfy the equations of motion of the type (1.4) - (1.6).

We note that the domains defined above include in particular the following ones:

a) a parallelepiped, one parallel pair of faces of which moves in the relevant axial direction according to (1.4), the second pair according to (1.5) and the third pair according to (1.6);

b) a bounded hollow cylinder whose side surfaces follow (2.4) and the end-walls any one of (1.4), (1.5) or (1.6).

3. Obviously, when the domain consists of a spherical symmetrical layer bounded by spherical surfaces, the equations of motion of which coincide with either (1.4) or (1.5) or (1.6), application of the same procedure yields an exact solution of the initial problem in spherical coordinates, as the substitution $W = U/r$ (r is the coordinate and W is the new function) transforms the equation

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial U}{\partial r} = \frac{\partial U}{\partial t} + f(r, t)$$

into (1.1) which has been already considered.

BIBLIOGRAPHY

1. Grinberg G. A., Solution of diffusion-type problems for expanding or contracting regions. PMM Vol. 33, No. 2, 1969.

2. Grinberg G. A. and Koss V. A., Solution of diffusion type problems for expanding or contracting regions whose form varies with time without preserving similarity. PMM Vol. 33, No. 4, 1969.

3. Grinberg G. A., On the temperature or concentration fields produced inside an infinite or finite domain by moving surfaces at which the temperature or concentration are given as functions of time, PMM Vol. 33, No. 6, 1969.

Translated by L. K.